UNIVERSITI SAINS MALAYSIA

First Semester Examination
Academic Session 2015/2016

December 2015/January 2016

MSG 327 – Mathematical Modelling
[Pemodelan Matematik]

Duration : 3 hours
[Masa : 3 jam]

Please check that this examination paper consists of SEVEN pages of printed materials before you begin the examination.

[Sila pastikan bahawa kertas peperiksaan ini mengandungi TUJUH muka surat yang bercetak sebelum anda memulakan peperiksaan ini.]

Instructions: Answer **THREE** (3) questions.

*[**Arahan:** Jawab **TIGA** (3) soalan.]*

In the event of any discrepancies, the English version shall be used.

[Sekiranya terdapat sebarang percanggahan pada soalan peperiksaan, versi Bahasa Inggeris hendaklah diguna pakai].

1. Consider the following advection-dispersion-daecay equation (1) at steady state

$$E\frac{\partial^2 c}{\partial x^2} - u\frac{\partial c}{\partial x} - k\,c = 0 \qquad (1)$$

with the boundary conditions c = 0 kg/m^3 when $x = -\infty$ and $x = +\infty$. Suppose a factory discharges a chemical into the river at x = 0 m at the rate W kg/s and mixed uniformly. Therefore, at this discharge point, x = 0 m, the chemical concentration will be a maximum (c_0).

(a) Assume that the dispersion coefficient E m^2/s, the velocity u m/s and the chemical decay rate k s^{-1} are constant. Show that the analytical solution to (1) is

$$c(x) = \begin{cases} c_0 e^{\frac{u}{2E}(1+\alpha)x} & x \le 0 \\ c_0 e^{\frac{u}{2E}(1-\alpha)x} & x \ge 0 \end{cases}$$

with $\alpha = \sqrt{1 + \frac{4kE}{u^2}} > 1$.

(b) Show that $c_0 = \frac{W}{Q\alpha}$ kg/m^3 with $Q = uA$ m^3/s in which A m^2 is the cross sectional area of the river.

(c) Suppose W = 0.1 kg/s, u = 0.03 m/s, E = 10 m^2/s, $k = 10^{-4}$ s^{-1} and A = 10 m^2. Find the chemical concentration $c(x)$ in the river at x = –500 m, –200 m, 200 m, 500 m and 1000 m. Sketch the graph of $c(x)$ for $x \in (-\infty, \infty)$.

[100 marks]

1. *Pertimbangkan persamaan adveksi-sebaran-pereputan (1) berikut pada keadaan mantap*

$$E\frac{\partial^2 c}{\partial x^2} - u\frac{\partial c}{\partial x} - k\,c = 0 \qquad (1)$$

dengan syarat-syarat sempadan c = 0 kg/m³ apabila x = –∞ and x = +∞. Andaikan satu kilang melepaskan suatu bahan kimia ke dalam sungai pada x = 0 m pada kadar W kg/s dan dicampur seragam. Maka, pada titik pelepasan ini, x = 0 m, kepekatan bahan kimia tersebut akan menjadi maksimum (c_0 *).*

(a) *Andaikan bahawa pekali sebaran E* m^2/s*, halaju u m/s dan kadar pereputan bahan kimia k* s^{-1} *adalah malar. Tunjukkan bahawa penyelesaian analitikal untuk (1) ialah*

$$c(x) = \begin{cases} c_0 e^{\frac{u}{2E}(1+\alpha)x} & x \le 0 \\ c_0 e^{\frac{u}{2E}(1-\alpha)x} & x \ge 0 \end{cases}$$

dengan $\alpha = \sqrt{1 + \frac{4kE}{u^2}} > 1.$

(b) *Tunjukkan bahawa* $c_0 = \frac{W}{Q\alpha}$ *kg/m³ dengan Q = uA* m^3/s *di mana A ialah luas keratan rentas sungai.*

(c) *Andaikan W = 0.1 kg/s, u = 0.03 m/s, E = 10* m^2/s*, k =* 10^{-4} s^{-1} *dan A = 10* m^2*. Cari kepekatan bahan kimia c(x) dalam sungai pada x = –500 m, –200 m, 200 , 500 m dan 1000 m. Lakarkan graf c(x) untuk* $x \in (-\infty, \infty)$*.*

[100 markah]

2. Equation (1) can be solved numerically by using finite segment method to obtain the chemical concentration in a uniform river. Suppose that a uniform river has cross-sectional area A = 10 m^2, length L = 1500 m, velocity u = 0.03 ms^{-1} and dispersion coefficient E = 10 m^{2}s^{-1}. This river is divided into three uniform segments with Δx = 500 m for each segment as shown in Figure 1. A chemical is released into segment 1 of the river at the rate of W = 0.1 kg s^{-1}. This chemical decays at the rate of $k = 10^{-4}$ s^{-1}. The water at the upstream (c_u) and downstream (c_d) segments is assumed to have a chemical concentration of 0.0 kg m^{-3}.

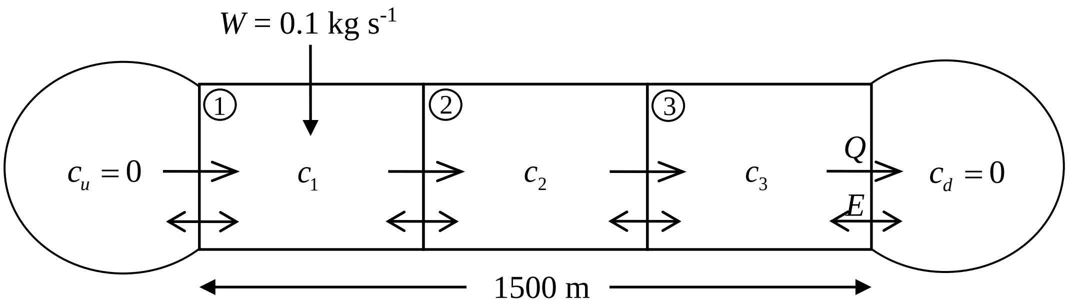


Figure 1. A uniform river divided into three segments (c_1, c_2, c_3) with upstream segment (c_u) and downstream segment (c_d)

(a) Find the concentration for every segment c_1, c_2, c_3 after steady state is achieved.

(b) Suppose that a lake represented by segment 4 is added to the system as shown in Figure 2.

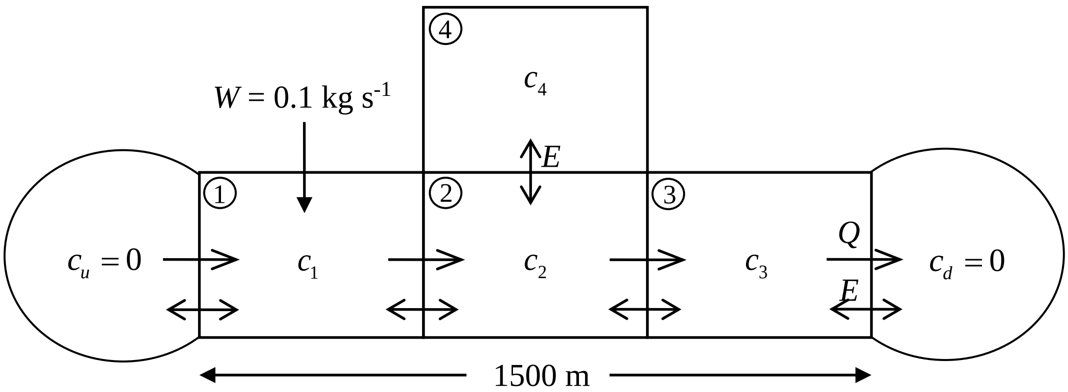


Figure 2. A uniform river divided into three segments (c_1, c_2, c_3) with upstream and downstream segments (c_u and c_d) and lake segment (c_4)

(i) Find the concentration c_4 at Segment 4 after steady state is achieved.

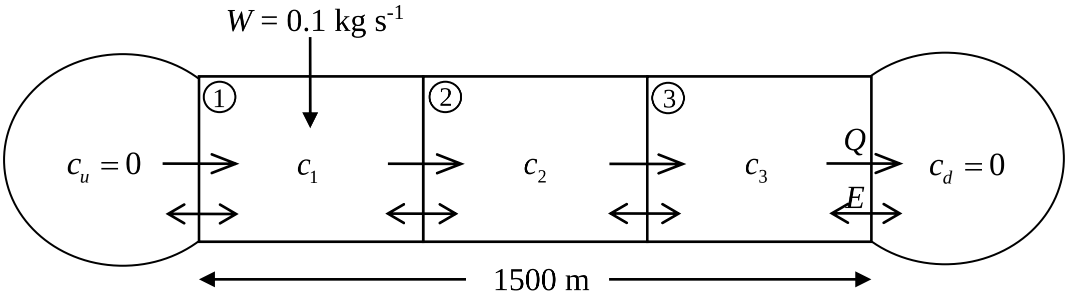
(ii) Find and discuss the concentration at Segment 4 when $E_{24} \to \infty$. Here, E_{24} represents the dispersion between Segment 2 and Segment 4.

(iii) Find and discuss the concentration at Segment 4 when $E_{24} \to 0$.

[100 marks]

...5/-

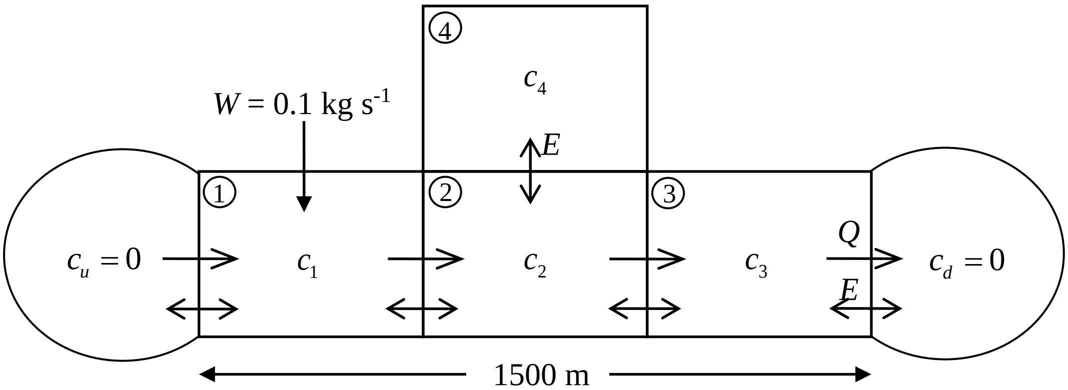
2. *Persamaan (1) boleh diselesaikan secara berangka dengan menggunakan kaedah segmen terhingga untuk mendapatkan kepekatan kimia di dalam sungai yang seragam. Andaikan satu sungai seragam mempunyai luas keratan rentas $A = 10$ m^2, panjang $L = 1500$ m, halaju $u = 0.03$ ms^{-1} dan pekali sebaran $E = 10$ m^2s^{-1}. Sungai ini dibahagikan kepada tiga segmen seragam dengan $\Delta x = 500$ m bagi setiap segmen seperti yang ditunjukkan dalam Rajah 1. Suatu bahan kimia dilepaskan ke dalam segmen 1 sungai tersebut pada kadar $W = 0.1$ kg s^{-1}. Bahan kimia ini mereput pada kadar $k = 10^{-4}$ s^{-1}. Air di segmen hulu (c_u) dan segmen hilir (c_d) dianggap mempunyai kepekatan kimia 0.0 kg m^{-3}.*



Rajah 1. Sungai seragam dibahagikan kepada tiga segmen (c_1, c_2, c_3) dengan segmen hulu (c_u) dan segmen hilir (c_d)

(a) *Cari kepekatan untuk setiap segmen c_1, c_2, c_3 selepas keadaan mantap dicapai.*

(b) *Andaikan satu tasik yang diwakili oleh Segmen 4 ditambah ke sistem tersebut seperti yang ditunjukkan dalam Rajah 2.*



Rajah 2. Sungai seragam dibahagikan kepada tiga segmen (c_1, c_2, c_3) dengan segmen hulu dan hilir (c_u, c_d) dan segmen tasik (c_4)

(i) *Cari kepekatan c_4 di segmen 4 selepas keadaan mantap dicapai.*

(ii) *Cari dan bincangkan kepekatan pada Segmen 4 apabila $E_{24} \to \infty$. Di sini, E_{24} mewakili sebaran antara Segmen 2 dan Segmen 4.*

(iii) *Cari dan bincangkan kepekatan pada Segmen 4 apabila $E_{24} \to 0$.*

[100 markah]

3. Let BOD concentration in a fish tank at time t day be denoted by $\ell(t)$ mg/L and DO concentration in the fish tank at time t be denoted by $c(t)$ mg/L. The initial concentrations of BOD and DO in that tank are denoted by ℓ_0 mg/L and c_0 mg/L, respectively. BOD decay rate, α day^{-1}, DO reaeration rate, β day^{-1} and DO saturation concentration, c_s mg/L are assumed constants. Any additional symbols used must be defined, with proper dimension and unit.

(a) Write the differential equations to describe the changes of BOD and DO concentrations with respect to time in a uniformly-mixed *closed* tank. Then find the solutions of the differential equations. Sketch the time series for BOD and DO concentrations in the *closed* tank when:

(i) $c_0 > \ell_0$

(ii) $c_0 < \ell_0$

(iii) $c_0 = \ell_0$

(b) Now consider the case with the tank opened so that air can enter the tank to reaerate the water. Write the differential equations to describe the changes of BOD and DO concentrations with respect to time in this *open* tank. Then find the solutions of the differential equations. Sketch the time series for BOD and DO concentrations in the *open* tank when:

(i) $c_0 > \ell_0 > c_s$

(ii) $c_0 < \ell_0 < c_s$

(c) Suppose the open tank has pure water (c_{in} = 8 mg/L) flowing into it at the rate of Q m^3/day. If the BOD concentration in the open tank were to remain constant at ℓ_c mg/L and the volume of water in the tank were to remain constant at V m^3, derive the steady state DO concentration in the tank. Let $\ell_c = 6$ mg/L, $c_0 = 2$ mg/L, α = 0.3 day^{-1}, β = 0.6 day^{-1}, $c_s = 7$ mg/L and V = 1 m^3. Calculate the required flow Q m^3/day to maintain a constant DO concentration of 5 mg/L in the tank.

[100 marks]

3. *Biarkan kepekatan BOD dalam suatu tangki ikan pada masa t hari diwakili oleh* $\ell(t)$ *mg/L dan kepekatan DO dalam tangki ikan tersebut pada masa t diwakili oleh* $c(t)$ *mg/L. Kepekatan awal BOD dan DO dalam tangki tersebut masing-masing diwakili oleh* ℓ_0 *mg/L dan* c_0 *mg/L. Kadar pereputan BOD,* α *hari*$^{-1}$*, kadar pengudaraan semula DO,* β *hari*$^{-1}$ *dan kepekatan tepu DO,* c_s *mg/L dianggap malar. Simbol tambahan yang digunakan mestilah didefinisikan, dengan dimensi dan unit yang betul.*

   (a) *Tulis persamaan pembezaan untuk menggambarkan perubahan kepekatan BOD dan DO terhadap masa dalam satu tangki tertutup yang bercampur seragam. Kemudian cari penyelesaian bagi persamaan pembezaan tersebut. Lakarkan siri masa bagi kepekatan BOD dan DO dalam tangki tertutup tersebut apabila:*

      (i) $c_0 > \ell_0$

      (ii) $c_0 < \ell_0$

      (iii) $c_0 = \ell_0$

   (b) *Sekarang pertimbangkan kes dengan tangki terbuka supaya udara dapat masuk tangki untuk pengudaraan semula. Tulis persamaan pembezaan untuk menggambarkan perubahan kepekatan BOD dan DO terhadap masa dalam tangki terbuka ini. Kemudian cari penyelesaian bagi persamaan pembezaan tersebut. Lakarkan siri masa bagi kepekatan BOD dan DO dalam tangki terbuka tersebut apabila:*

      (i) $c_0 > \ell_0 > c_s$

      (ii) $c_0 < \ell_0 < c_s$

   (c) *Andaikan tangki terbuka tersebut mempunyai air bersih (*c_{in} *= 8 mg/L) mengalir ke dalamnya pada kadar Q m*^{3}*/hari. Jika kepekatan BOD dalam tangki tersebut dikekal malar pada* ℓ_c *mg/L dan isipadu air dalam tangki tersebut dikekal malar pada V m*^{3}*, terbitkan kepekatan DO dalam tangki tersebut pada keadaan mantap. Biar* $\ell_c = 6$ *mg/L,* $c_0 = 2$ *mg/L,* α *= 0.3 hari*$^{-1}$*,* β *= 0.6 hari*$^{-1}$*,* $c_s = 7$ *mg/L dan V = 1 m*^{3}*. Kira aliran Q m*^{3}*/hari yang diperlukan untuk mengekalkan kepekatan DO dalam tangki pada 5 mg/L.*

[100 markah]

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